1/

Math 112 : Introductory Real Analysis

§ Differentiation

Def Let f be a real-valued function on $[a,b]$.

For any $x \in [a,b]$, define the *derivative of f at x* to be

$$f'(x) := \lim_{t \to x} \frac{f(t)-f(x)}{t-x}$$

provided the limit exists, in which case we say f is *differentiable at x*.

Thm Let f be a real function on $[a,b]$.

If f is differentiable at $x \in [a,b]$, then f is continuous at x.

proof)

$$\lim_{t \to x} \left(f(t)-f(x)\right) = \lim_{t \to x} (t-x) \cdot \frac{f(t)-f(x)}{t-x} = 0 \cdot f'(x) = 0 \quad \blacksquare$$

So, differentiability is stronger than continuity!

Thm Suppose f and g are real functions on $[a,b]$ which are differentiable at $x \in [a,b]$. Then

(a) $(f+g)'(x) = f'(x) + g'(x)$

(b) $(fg)'(x) = f'(x)g(x) + f(x)g'(x)$

(c) $\left(\frac{f}{g}\right)'(x) = \dfrac{g(x)f'(x) - g'(x)f(x)}{g^2(x)}$ assuming that $g(x) \neq 0$

2/

proof)

(a) $$\lim_{t\to x}\frac{(f+g)(t)-(f+g)(x)}{t-x} = \lim_{t\to x}\left(\frac{f(t)-f(x)}{t-x}+\frac{g(t)-g(x)}{t-x}\right) = f'(x)+g'(x).$$

(b) $$\lim_{t\to x}\frac{(fg)(t)-(fg)(x)}{t-x} = \lim_{t\to x}\frac{f(t)\,(g(t)-g(x)) + (f(t)-f(x))\,g(x)}{t-x}$$
$$= f(x)\,g'(x) + f'(x)\,g(x).$$

(c) $$\lim_{t\to x}\frac{\left(\frac{f}{g}\right)(t)-\left(\frac{f}{g}\right)(x)}{t-x} = \lim_{t\to x}\frac{1}{g(t)g(x)}\left(g(x)\frac{f(t)-f(x)}{t-x} - \frac{g(t)-g(x)}{t-x}f(x)\right)$$
$$= \frac{1}{g^2(x)}\left(g(x)f'(x)-g'(x)f(x)\right). \quad \blacksquare$$

Examples

- $f(x)=c$ ($c\in\mathbb{R}$ constant) $\Rightarrow f'(x)=0$
- $f(x)=x \Rightarrow f'(x)=1$
- $f(x)=x^n \Rightarrow f'(x)=nx^{n-1}$ (In case $n<0$, assuming $x\neq 0$)
  ($n\in\mathbb{Z}$)

★ Thm (Chain rule) Suppose f is ~~continuous~~ a real function on $[a,b]$, $f'(x)$ exists at some point $x\in[a,b]$, g is defined on an interval I which contains $f([a,b])$, and $g'(f(x))$ exists.

If $h(t)=g(f(t))$ $(a\le t\le b)$, then h is differentiable at x, and
$$h'(x) = g'(f(x))\cdot f'(x).$$

3/ proof of the chain rule)

Let $y = f(x)$.

By the definition of the derivative, we have

$$f(t) - f(x) = (t-x)\big(f'(x) + u(t)\big),$$
$$g(s) - g(y) = (s-y)\big(g'(y) + v(s)\big),$$

where $t \in [a,b]$, $s \in I$, and $\lim_{t \to x} u(t) = 0$ and $\lim_{s \to y} v(s) = 0$.

Then, we obtain

$$\begin{aligned} h(t) - h(x) &= g(f(t)) - g(f(x)) \\ &= \big(f(t) - f(x)\big)\big(g'(f(x)) + v(f(t))\big) \\ &= (t-x)\,\big(f'(x) + u(t)\big)\,\big(g'(f(x)) + v(f(t))\big), \end{aligned}$$

~~or if $t \neq x$~~,

Hence

$$\begin{aligned} \lim_{t \to x} \frac{h(t) - h(x)}{t - x} &= \lim_{t \to x} \big(f'(x) + u(t)\big)\big(g'(f(x)) + v(f(t))\big) \\ &= f'(x) \cdot g'(f(x)). \quad \blacksquare \end{aligned}$$

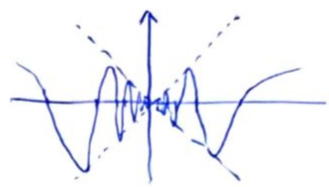
by continuity of f at x

Examples

(a) Let f be defined by

$$f(x) = \begin{cases} x \sin \frac{1}{x} & (x \neq 0), \\ 0 & (x = 0). \end{cases}$$

f is continuous at every point



Taking for granted that $\sin' x = \cos x$,

we obtain $f'(x) = \sin\frac{1}{x} + x\cos\frac{1}{x} \cdot \left(-\frac{1}{x^2}\right)$

$$= \sin\frac{1}{x} - \frac{1}{x}\cos\frac{1}{x} \quad (x \neq 0).$$

At $x = 0$, $\dfrac{f(t) - f(0)}{t - 0} = \sin\frac{1}{t}$, and as $t \to 0$, the limit does not exist;

i.e. $f'(0)$ does not exist.

(b) Let f be defined by

$$f(x) = \begin{cases} x^2 \sin \frac{1}{x} & (x \neq 0), \\ 0 & (x = 0). \end{cases}$$

We obtain $f'(x) = 2x \sin\frac{1}{x} - \cos\frac{1}{x} \quad (x \neq 0)$.

At $x = 0$, $\left|\dfrac{f(t) - f(0)}{t - 0}\right| = \left|t \sin\frac{1}{t}\right| \leq |t| \quad (t \neq 0)$,

and hence $f'(0) = \lim\limits_{t \to 0} \dfrac{f(t) - f(0)}{t - 0} = 0$.

Thus f is differentiable at every point, but f' is not a continuous function;

it has a discontinuity of the second kind at 0.

(We'll see later that for any differentiable f, f' cannot have any simple discontinuities.)

5/

- Mean value theorems

<u>Def</u> Let f be a real function on a metric space X.

We say that f has a local maximum at $p \in X$

if there exists $\delta > 0$ such that $f(q) \leq f(p)$ for all $q \in B_\delta(p)$.

Local minima are defined likewise.

<u>Thm</u> Let f be a real function on $[a,b]$.

If f has a local maximum at $x \in (a,b)$ and if $f'(x)$ exists,

then $f'(x) = 0$.

proof) Choose $\delta > 0$ so that $f(t) \leq f(x)$ for all $t \in (x-\delta, x+\delta)$.

Then $\dfrac{f(t)-f(x)}{t-x} \geq 0$ for all $t \in (x-\delta, x)$.

Letting $t \to x$, we see that $f'(x) \geq 0$.

Likewise, $\dfrac{f(t)-f(x)}{t-x} \leq 0$ for all $t \in (x, x+\delta)$,

and letting $t \to x$, we see that $f'(x) \leq 0$.

Hence $f'(x) = 0$. ■

<u>Thm</u> (Generalized mean value theorem) Let f and g be continuous real functions on $[a,b]$ which are differentiable in (a,b).

Then there is a point $x \in (a,b)$ such that

$$(f(b) - f(a))\, g'(x) = (g(b) - g(a))\, f'(x).$$

proof) Put $h(t) = (f(b)-f(a))g(t) - (g(b)-g(a))f(t)$ $(a \leq t \leq b)$.

Then h is continuous on $[a,b]$ and differentiable in (a,b).

Moreover, $h(a) = f(b)g(a) - f(a)g(b) = h(b)$.

We need to show that $h'(x)=0$ for some $x \in (a,b)$.

By the extreme value theorem, h attains its maximum and minimum, and since $h(a)=h(b)$, there is some $x \in (a,b)$ where f has either a maximum or a minimum.

By the previous theorem, $h'(x)=0$. ■

Cor (Mean value theorem) If f is a real function on $[a,b]$ which is differentiable in (a,b), then there is a point $x \in (a,b)$ at which

$$f(b)-f(a) = (b-a)f'(x).$$

proof) Take $g(x)=x$ in the previous theorem. ■

Thm Let f be a real differentiable function on $[a,b]$.
If $f'(a) < \lambda < f'(b)$, then there is a point $x \in (a,b)$ such that $f'(x)=\lambda$.

proof) Put $g(t) = f(t) - \lambda t$. Then $g'(a)<0$ and $g'(b)>0$.

Hence, by the extreme value theorem, g attains its minimum at some $x \in (a,b)$.

Thus $g'(x)=0$, i.e. $f'(x)=\lambda$. ■

Cor If f is differentiable on $[a,b]$, then f' cannot have any simple discontinuities on $[a,b]$.